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Multifractal Height Cross-Correlation Analysis

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Abstract

We introduce a new method for detection of long-range cross-correlations and cross-multifractality – multifractal height cross-correlation analysis (MF-HXA). We show that long-range cross-correlations can be caused by long-range dependence of separate processes and the correlations above them. Similar separation applies for cross-multifractality – standard separation between distributional properties and correlations is enriched by division of correlations between auto-correlations and cross-correlations. Efficiency of the method is showed on two types of simulated series – ARFIMA and Mandelbrot's Binomial Multifractal model. We further apply the method on returns and volatility of NASDAQ and S&P500 indices and uncover some interesting results.

Keywords: multifractality, long-range dependence, cross-correlations

JEL codes: C4, C5, G12

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1 Introduction

Research of long-range dependence and multifractality in various time series has grown significantly during the last years, e.g. Di Matteo (2007); Matos et al. (2008); Czarnecki et al. (2008); Grech and Mazur (2004). Efficient detection of long-range dependence and estimation of Hurst exponent is crucial for financial analysts as its presence has important implications for portfolio selection, option pricing and risk management. There are several methods for long-range dependence detection, among the most popular are rescaled range analysis (Hurst, 1951), modified rescaled range analysis (Lo, 1991), rescaled variance analysis

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(Giraitis et al., 2003), detrended fluctuation analysis (Peng et al., 1994) and detrending moving average (Alessio et al., 2002). For detection of multifractality, there are three popular methods – multifractal detrended fluctuation analysis (MF-DFA) of Kantelhardt et al. (2002), generalized Hurst exponent approach (GHE) of Di Matteo et al. (2003), which is based on height-height correlation analysis of Barabasi et al. (1991), and wavelet transform modulus maxima (WTMM) of Muzy et al. (1991). The precision of various methods has been discussed as well (Couillard and Davison, 2005; Grech and Mazur, 2005; Weron, 2002; Barunik and Kristoufek, 2010; Kristoufek, 2010).

Recently, the examination of long-range cross-correlations has become of an interest as it provides more information about the process. Podobnik and Stanley (2008) generalized detrended fluctuation analysis for two time series and introduced detrended cross-correlation analysis (DCCA). Zhou (2008) further generalized the method and introduced multifractal detrended cross-correlation analysis. In this paper, we introduce two new methods, which are a generalization of of height-height correlations of Barabasi et al. (1991) – multifractal height cross-correlation analysis (MF-HXA) and its special case of height cross-correlaction analysis (HXA).

The paper is structured as follows. In Section 2, we briefly discuss the basic notions of long-range correlations and multifractality. Section 3 introduces the method of MF-HXA and discusses long-range cross-correlations and cross-multifractality in detail. In Section 4, we show the efficiency of the method on two simulated types of processes. In Section 5, we apply MF-HXA on daily returns and volatility of NASDAQ and S&P500 between 1.1.2000 and 31.12.2009. We find that there are significant cross-correlations in the process of volatilities which is mainly caused by the long memory in the separate processes. Moreover, multifractal analysis reveals that the correlations are the highest for the extreme events whereas the multifractality remains rather weak. Section 6 concludes.

2 Long-range correlations and multifractality

In this section, we present basic notions of multifractality, long-range correlations and long-range cross-correlations. As the subject is widely discussed in the recent literature, we present only brief description. For more detailed reviews, see Beran (1994); Kantelhardt (2009); Embrechts and Maejima (2002).

Stationary process is long-range dependent if autocorrelation function ρ of said process decays as $\rho(k) \approx Ck^{2H-2}$ for lag $k \to \infty$. Parameter $0 < H < 1$ is called Hurst exponent after water engineer Harold Edwin Hurst who used the exponent for description of river flows behavior of the Nile River (Hurst, 1951; Mandelbrot and van Ness, 1968).

A critical value of Hurst exponent is 0.5 and suggests two possible processes – either an independent process (Beran, 1994) or a short-term dependent process (Lillo and Farmer, 2004). If $H > 0.5$, auto-covariances decay hyperbolically and are positive at all lags, the process is then called long-range dependent with positive correlations (Embrechts and Maejima, 2002) or persistent (Mandelbrot and van Ness, 1968). On the other hand, if $H < 0.5$, auto-covariances again decay hyperbolically and are negative at all lags and the process is said to be long-range dependent with negative correlations (Embrechts and Maejima, 2002) or anti-persistent (Mandelbrot and van Ness, 1968). The persistent process implies that a positive movement is statistically more likely to be followed by another positive movement or vice versa. On the other hand, the anti-persistent process implies that a positive movement is more statistically probable to be followed by a negative movement and vice versa (Vandewalle et al., 1997).

If the process can be described by single Hurst exponent H, it is called monofractal. If different Hurst exponents are needed for various scales, the process exhibits crossovers. Further, there can be different Hurst exponents for parts of the series, which is solved by a use of time-dependent (or local) Hurst exponent (Grech and Mazur, 2004). The most complicated is the case when there is a whole spectrum of Hurst exponents which is needed for a full description of the process made of many complex fractal processes (Kantelhardt et al., 2002).

Both of the above described phenomena can be present in the relation between two separate series. A series may be long-range dependent but can also have a long memory of a different process so that it is long-range cross-dependent with Hurst exponent H_{xy}. Cross-correlation function ρ_{xy} of processes x_t and y_t then decays as $\rho_{xy} \approx Ck^{2H_{xy}-2}$. Similarly to the standard case, if the whole spectrum of cross-correlation Hurst exponents H_{xy} is needed for description of cross-correlations between two processes, the relation is cross-multifractal. Further features of long-range cross-correlations and cross-multifractality are discussed in following sections.

3 Multifractal height cross-correlation analysis

We introduce the multifractal height cross-correlation analysis (MF-HXA) in this section. The connection to the generalized Hurst exponent approach (GHE) is discussed in detail as well as a crucial division of long-range cross-correlations. The last subsection discusses a detection of cross-multifractality in a pair of series.

3.1 Method

Detection of long-range dependence and estimation of generalized Hurst exponent $H(q)$ of Barabasi et al. (1991) is based on q-th order height-height correlation function of time series $X(t)$, with $q > 0$, as

$$K_q(\tau) = \frac{1}{T-\tau} \sum_{t=0}^{T-\tau} (|X(t+\tau) - X(t)|^q) \tag{1}$$

which scales as

$$K_q(\tau) \approx c\tau^{qH(q)}. \tag{2}$$

We generalize the method presented above and introduce the multifractal height cross-correlation analysis (MF-HXA) which can be used for the detection of long-range correlations and multifractality between two separate time series.

In the procedure, we take the first differences of time series $\{x(t)\}_{t=0}^{T}$ and $\{y(t)\}_{t=0}^{T}$ and obtain $\{\triangle x(t)\}_{t=1}^{T}$ and $\{\triangle y(t)\}_{t=1}^{T}$. The differences are further standardized by deduction of the corresponding mean μ and division by the corresponding standard deviation σ so that we get new series $\{\triangle\tilde{x}(t)\}_{t=1}^{T} = \{\frac{\triangle x(t)-\mu_x}{\sigma_x}\}_{t=1}^{T}$ and $\{\triangle\tilde{y}(t)\}_{t=1}^{T} = \{\frac{\triangle y(t)-\mu_y}{\sigma_y}\}_{t=1}^{T}$. Further, the series are cumulated so that we obtain $\{X(t)\}_{t=0}^{T}$ and $\{Y(t)\}_{t=0}^{T}$ with $X(t) = \sum_{i=1}^{t}\triangle\tilde{x}(i)$ and $Y(t) = \sum_{i=1}^{t}\triangle\tilde{y}(i)$. Moreover, $X(0) = Y(0) = 0$. Generalizing Equation 1 for two time series, we obtain

$$K_{xy,q}(\tau) = \frac{1}{T-\tau}\sum_{t=0}^{T-\tau}(|[X(t+\tau)-X(t)][Y(t+\tau)-Y(t)]|^{q/2}) \tag{3}$$

For $q = 1$, generalized height correlation function represents the scaling of absolute deviations of covariates, and for $q = 2$, it corresponds to standard cross-correlation function. We propose multifractal height cross-correlation analysis (MF-HXA) based on the generalization of Equation 2. Scaling relationship between $K_{xy,q}(\tau)$ and generalized cross-correlation Hurst exponent $H_{xy}(q)$ is obtained as

$$K_{xy,q}(\tau) \approx c\tau^{qH_{xy}(q)}. \tag{4}$$

For $q = 2$, the method can be used for the detection of long-range cross-correlations solely and we call it height cross-correlation analysis (HXA). Obviously, for $\{x(t)\}_{t=0}^{T} = \{y(t)\}_{t=0}^{T}$, MF-HXA turns into the generalized Hurst exponent approach of Di Matteo et al. (2003), which is equivalent to height-height correlation analysis of Barabasi et al. (1991).

3.2 Two types of cross-corelations

Similarly to Hurst exponent $H(2)$, the cross-correlation Hurst exponent $0 < H_{xy}(2) < 1$ has a critical value of 0.5 which indicates that the examined series are cross-independent (or cross-short-range-dependent). For $H_{xy}(2) > 0.5$, the series are cross-persistent so that an increment (a decrement) in $\triangle x(t)\triangle y(t)$ is more statistically probable to be followed by another increment (decrement) in $\triangle x(t+1)\triangle y(t+1)$. Reversely for $H_{xy}(2) < 0.5$, the series are cross-antipersistent so that an increment (a decrement) in $\triangle x(t)\triangle y(t)$ is more statistically probable to be followed by a decrement (an increment) in $\triangle x(t+1)\triangle y(t+1)$.

For cross-independent and cross-short-range-dependent processes, the cross-correlations at high lags are not significant so that $[X(t+\tau)-X(t)][Y(t+\tau)-Y(t)] \approx 0$. If $T \gg \tau$, it can be easily shown that

$$H_{xy}(q) \approx \frac{H_{xx}(q)+H_{yy}(q)}{2} \tag{5}$$

Therefore, $H_{xy}(2) \neq 0.5$ can be caused by long-range dependence of the two processes even if there are no "true" long-range cross-correlations. Therefore, we need to distinguish between two types of long-range cross-correlations: (i) primary long-range cross-correlations caused by true long-range interrelation between two series where the series is not only affected by its own long-range dependence but also by the long memory of the other series, and (ii) secondary long-range cross-correlations which are caused by long-range dependence of the separate series. To distinguish between the two types of cross-correlations, we can use Equation 5.

3.3 Cross-multifractality

If a spectrum of Hurst exponents $H_{xy}(q)$ is needed to describe the relationship between two time series, the series are cross-multifractal. The generalized cross-correlation Hurst exponent $H_{xy}(q)$ is independent of q for monofractal series or it is dependent on q for multifractal series. For high qs, the averages in Equation 3 are dominated by high values of $[X(t+\tau)-X(t)][Y(t+\tau)-Y(t)]$; for low qs, the averages are dominated by low values of $[X(t+\tau)-X(t)][Y(t+\tau)-Y(t)]$. Therefore, $H_{xy}(q)$ is expected to be decreasing in q. Moreover, the influence of joint distributional properties implies that multifractality can be due to the cross-correlations as well as the broadness of the joint-distribution (Kantelhardt, 2009). Again, the effect of correlations can be separated into two – auto-correlations and cross-correlations according to Equation 5.

To examine a scale of multifractality, there are two measures of multifractality usually used – α and $f(\alpha)$. These are used as an additional tool to a simple examination of the behavior of $H_{xy}(q)$ as the generalized exponents can vary significantly even for monofractal processes (Kantelhardt et al., 2002). Both measures are partially connected to the scaling exponent $\tau(q)$, which is defined as $\tau(q) = qH(q)$ for a standard case and as $\tau(q)_{xy} = qH_{xy}(q)$ from Equations 2 and 4. Singularity strength, or Hölder exponent, α is a characteristic measure of a series whereas singularity spectrum $f(\alpha)$ characterizes a dimension of a series characterized by α. To obtain α and $f(\alpha)$, we generalize the procedure of Barabasi et al. (1991) for two time series.

To characterize the relationship between the series $X(t)$ and $Y(t)$, we construct the a probability measure $p_t(\tau)$ connected to a hierarchy of changes of the two series. The measure is calculated as

$$p_{xy,t}(\tau) = \frac{\sqrt{|[X(t+\tau)-X(t)][Y(t+\tau)-Y(t)]|}}{\sum_{t=1}^{T-\tau}\sqrt{|[X(t+\tau)-X(t)][Y(t+\tau)-Y(t)]|}}. \tag{6}$$

As $p_{xy,t}(\tau)$ is a standard probability measure, it holds that $\sum_{t=1}^{T-\tau} p_{xy,t}(\tau) = 1$ and $p_{xy,t}(\tau) \geq 0$. We further define a generating function for two time series $\chi_{q,xy}(\tau)$, which is associated with the probability measures $p_{xy,t}(\tau)$, and corresponding generalized dimensions $D_{xy,q}$ as

$$\chi_{xy,q}(\tau) = \sum_{t=1}^{T-\tau} p_{xy,t}^{q}(\tau). \tag{7}$$

$$\chi_{xy,q}(\tau) = \tau^{(q-1)D_{xy,q}}. \tag{8}$$

Finally, we use Legendre transformation and obtain singularity strength α through change of generalized dimension $D_{xy,q}$ with varying q. Singularity spectrum $f(\alpha)$ is then obtained with a use of both α and $D_{xy,q}$. The specific relationships hold as follows

$$\alpha_{xy} = \frac{\partial[(q-1)D_{xy,q}]}{\partial q} \tag{9}$$

$$f(\alpha_{xy}) = q\alpha_{xy} - (q-1)D_{xy,q} \tag{10}$$

The above described lengthy procedure can be alternatively replaced by a more simple one. If we assume that the probability measure $p_{xy,t}(\tau)$ describes the hierarchy of both series $X(t)$ and $Y(t)$ uniformly, we can write $p_{xy,t}(\tau) = \frac{1}{T}$ for $\tau \to 0$. Such assumption allows to use only $H_{xy}(q)$ for the construction of singularity strength α and singularity spectrum $f(\alpha)$. It then holds that[1]

$$\alpha_{xy} = \frac{\partial[qH_{xy}(q)]}{\partial q} - H_{xy}(1) \tag{11}$$

$$f(\alpha_{xy}) = q\frac{\partial[qH_{xy}(q)]}{\partial q} - qH_{xy}(q) \tag{12}$$

Note that for each of Equation 9 – 12, a unity is sometimes added to both values of α and $f(\alpha)$ in the literature dealing with multifractal spectrum and singularities (Kantelhardt et al., 2002).

4 Two illustrative examples

To validate MF-HXA, we present results for two randomly generated processes – two independent ARFIMA processes and two independent multifractal series based on Mandelbrot's Binomial Multifractal model. Note that both variants of the processes are independent so that the expected cross-correlation Hurst exponents $H_{xy}(q)$ are equal to arithmetic means of $H_{xx}(q)$ and $H_{yy}(q)$ of the separate processes.

[1] For a detailed derivation, see the Appendix of Barabasi et al. (1991)

4.1 Two ARFIMA processes

Autoregressive fractionally integrated moving average models (ARFIMA) are generalization of autoregressive moving average models (ARMA) of Box and Jenkins (1970) which allow for long-range dependence. With a use of backshift operator B, ARFIMA models are represented by $(1-\sum_{i=1}^{p}\varphi_i B^i)(1-B)^d X_t = (1+\sum_{i=1}^{q}\theta_i B^i)\varepsilon_t$, where $(1-B)^d = \sum_{k=0}^{d}\frac{(-1)^k B^k \Gamma(d+1)}{\Gamma(k+1)\Gamma(d-k+1)}$ (see Baillie et al. (1996) for details). d is a fractional differencing parameter and it holds that $d = H - 0.5$.

In Figure 1a, we show estimates of $H_{xx}(q)$, $H_{yy}(q)$ and $H_{xy}(q)$ for two independent ARFIMA processes with $H_{xx} = 0.7$ and $H_{yy} = 0.9$ with $T = 1000$, $\tau_{min} = 2$, $\tau_{max} = 100$ and $q = 0.1, 0.2, \dots, 9.9, 10$. Even though the both series are monofractal, the generalized Hurst exponents range from $H_{xx}(0.1) = 0.7442$ to $H_{xx}(10) = 0.6579$ and from $H_{yy}(0.1) = 0.9281$ to $H_{yy}(10) = 0.8546$. The differences are due to finite sample size and emphasize a need for using α and $f(\alpha)$ as the measures of multifractality. Importantly, the estimates of the generalized Hurst exponents characterizing the long-range dependence solely are close to the expected values – $H_{xx}(2) = 0.7134$ and $H_{yy}(2) = 0.9035$. Further, the estimates of cross-correlation Hurst exponents $H_{xy}(q)$ satisfy the relation of Equation 5 with only small deviations holds for all qs.

4.2 Mandelbrot's Binomial Multifractal series

Mandelbrot's Binomial Multifractal (MBM) is the simplest multifractal measure (Mandelbrot et al., 1997). Let $m_0 > 0$, $m_1 > 0$ and $m_0 + m_1 = 1$ and let us work on interval [0,1]. In the first stage, the mass of 1 is divided into two subintervals [0,1/2] and [1/2,1], when there is m_0 in the first subinterval and m_1 in the second one. In following stage, each subinterval is again halved and its mass is divided between the smaller subintervals in ratio $m_0 : m_1$. After k stages, we obtain a series of 2^k values. Note that the values are deterministically given as there is no noise added in the simplest version of the method. For an interval $[z, z+2^{-k}]$, the value μ has a value of $\mu[z, z+2^{-k}] = m_0^{k\varphi_0} m_1^{k\varphi_1}$, where φ_0 and φ_1 stand for relative frequencies of numbers 0 and 1 in a binary development of $2^k z$, respectively.

In Figure 1b, we show estimates $H_{xx}(q)$, $H_{yy}(q)$ and $H_{xy}(q)$ for two independent MBM models with $m_0 = 0.2$ and $m_0 = 0.4$, respectively. We generated the series with 10 steps and obtained $T = 1024$ observations and kept other parameters the same so that MF-HXA is run with $\tau_{min} = 2$, $\tau_{max} = 100$ and $q = 0.1, 0.2, \dots, 9.9, 10$. The variation of $H_{xx}(q)$, $H_{yy}(q)$ and $H_{xy}(q)$ is much stronger than in the case of monofractal ARFIMA models. The values range from $H_{xx}(0.1) = 0.8645$ to $H_{xx}(10) = 0.4305$ and from $H_{yy}(0.1) = 0.8175$ to $H_{yy}(10) = 0.6819$ for the respective processes. Importantly, Equation 5 holds for all qs with only insignificant deviations.

5 Application

To show the usefulness of MF-HXA, we apply the method on NASDAQ and S&P500 stock indices between 1.1.2000 and 31.12.2009 (2531 observations). The choice of the two indices from the same country has significant advantage over different variants as a number of different trading days is very limited and there is almost no need for data adjustments. We denote NASDAQ series as "x" and S&P500 series as "y" throughout the section. Evolution of logarithmic prices of both indices is shown in Figure 2 and basic descriptive statistics are summarized in Table 1. We can see that both indices move together whereas NASDAQ is more volatile. Both indices follow the standard stylized facts about financial markets with negatively skewed and leptokurtic returns (Cont, 2001).

We research on the potential long-range dependence and cross-correlations in returns and volatility. As a measure of volatility, we take absolute returns, which is standard in financial literature and also intuitive as returns can be taken as a product of a sign and a magnitude (absolute return). As one of the first indicators of the potential long-range correlations or cross-correlations, correlation functions are examined. In Figure 3a and 3b, there are auto-correlation and cross-correlation functions of returns and absolute returns, respectively, for lags $k = 1, 2, \ldots, 200$. Logarithmic representation of absolute values of correlation coefficients shows slow decay whereas correlations of returns show no such evolution. Even though the returns of the indices are strongly correlated with $\rho_{xy}(0) = 0.8652$, there are no significant correlations for higher lags. On the other hand, the absolute returns are less correlated than standard returns with $\rho_{xy}(0) = 0.7869$ but the long-range correlations and cross-correlations decay slowly indicating a potential presence of long-range dependence.

To test such assertion, we apply MF-HXA on both returns and volatility of NASDAQ and S&P500 with $T = 2531$, $\tau_{min} = 2$ and $\tau_{max} = 100$ for $q = 0.1, 0.2, \ldots, 9.9, 10$. We choose τ_{min} and τ_{max} to have enough values for the final regression according to Equation 4. A step of 0.1 of different qs ensures that the evolution of the generalized Hurst exponents, corresponding α and $f(\alpha)$ is smooth and well interpreted. Figures 4 and 5 show the estimates of $H_{xx}(q)$, $H_{yy}(q)$ and $H_{xy}(q)$ for returns and volatility, respectively. Figures 6 and 7 show the singularity strengths α and the singularity spectra $f(\alpha)$. To distinguish between different causes of multifractality, we also present the estimates for shuffled series in each of Figures 4 - 7. By shuffling, we tore all correlations in the data while the distribution remains the same. The multifractality in shuffled series is then caused by distribution solely.

From a simple graphical analysis, returns show only weak signs of multifractality while volatility seems more multifractal where the important part is caused by the distributional properties of the absolute returns. For NASDAQ, S&P500 and cross-multifractality, there are no significant differences. For more rigorous examination, we calculate differences between maximum and minimum generalized Hurst exponents for different qs and differences between maximum and minimum singularity strengths αs. The results are presented in Tables 2 and 3 for returns and volatility, respectively. In the Tables, relative differences represent percentages of the overall multifractality which is caused by correlations and the rest is due to distributional properties.

The results for returns are quite straightforward – the majority of multifractality in returns of NASDAQ is due to correlations whereas the opposite is true for S&P500. Cross-multifractality is caused by both cross-correlations and distribution broadness with equal weights. Note that the multifractality and cross-multifractality for returns is weak. For volatility, the majority of multifractality is caused by distributional properties where again for NASDAQ, the weight of correlations is higher than for S&P500. The case of cross-multifractality is very similar to the multifractality of NASDAQ. However, there is one significant difference. Note that such results can be read from both the generalized Hurst exponents and singularity strengths.

In Figure 5b, there is an obvious deviation of $H_{xy}(q)$ from the average of $H_{xx}(q)$ an $H_{yy}(q)$ for $q \geq 5$ while the deviation increases with q. Such behavior in shuffled series indicates that the scaling law is stronger in tails of joint-distribution which means that the series are more correlated in extreme events. Moreover, for the special case of $q = 2$, which corresponds to long-range dependence and cross-dependence, there is no significant dependence found in returns. However, there is very strong persistence found in volatility of both NASDAQ ($H_{xx}(2) = 0.9757$) and S&P500 ($H_{yy(2)} = 0.9728$) and strong cross-persistence ($H_{yy}(2) = 0.9781$) between the indices. However, when we compare $H_{xy}(2)$ for volatility with the average value of $H_{xx}(2)$ and $H_{yy}(2)$, we conclude that the cross-persistence is almost entirely due to the dependence in the separate indices.

6 Conclusions

In the paper, we introduced new method for detection of long-range cross-correlations and cross-multifractality – multifractal height cross-correlation analysis (MF-HXA). We showed that long-range cross-correlations can be caused by long-range dependence of separate processes and/or by dependence between the two series. Similarly for cross-multifractality, the causes can be separated into three groups – multifractality due to joint-distributional properties and due to correlations, which can be further divided into auto-correlations and cross-correlations.

We applied MF-HXA on returns and volatility of NASDAQ and S&P500 for the period between 1.1.2000 and 31.12.2009. We showed that there are no significant long-range correlations or cross-correlations in the returns. For volatility, both types of correlations are found, cross-correlations are found to be almost entirely due to strong persistence in separate processes of volatility. In multifractal analysis, we found interesting discrepancy between the two indices – majority of multifractality in returns is caused by correlations for NASDAQ and by distributional properties for S&P500. For volatility, multifractality is mainly caused by distributional properties. As for cross-multifractality, we uncovered strong correlations for extremal events whereas the cross-multifractality itself seems to be led more by NASDAQ component.

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Table 1: Descriptive statistics of NASDAQ and S&P500 returns

	NASDAQ	S&P500
Mean	-0.0002	-0.0001
SD	0.0181	0.0136
Skewness	-0.1269	-0.1262
Kurtosis	6.7762	11.5119
JB	1510	7644
JB (p-value)	0.0000	0.0000
correlation of returns	0.8652	0.8652
correlation of volatility	0.7869	0.7869

Table 2: Measures of multifractality for NASDAQ and S&P500 returns

	original	shuffled	difference	relative
$\max H_{xx} - \min H_{xx}$	0.1705	0.0401	0.1304	0.7649
$\max H_{yy} - \min H_{yy}$	0.1556	0.1432	0.0123	0.0793
$\max H_{xy} - \min H_{xy}$	0.1548	0.0773	0.0776	0.5010
$\max \alpha_{xx} - \min \alpha_{xx}$	3.0158	1.0205	1.9953	0.6616
$\max \alpha_{yy} - \min \alpha_{yy}$	3.3868	2.6387	0.7481	0.2209
$\max \alpha_{xy} - \min \alpha_{xy}$	3.1543	1.5841	1.5702	0.4978

Table 3: Measures of multifractality for NASDAQ and S&P500 volatility

	original	shuffled	difference	relative
$\max H_{xx} - \min H_{xx}$	0.1642	0.1139	0.0503	0.3065
$\max H_{yy} - \min H_{yy}$	0.1666	0.1169	0.0497	0.2982
$\max H_{xy} - \min H_{xy}$	0.1635	0.1010	0.0625	0.3822
$\max \alpha_{xx} - \min \alpha_{xx}$	3.7433	2.6796	1.0637	0.2842
$\max \alpha_{yy} - \min \alpha_{yy}$	2.4016	2.2525	0.1491	0.0621
$\max \alpha_{xy} - \min \alpha_{xy}$	2.8505	2.0745	0.7760	0.2722

(a) (b)

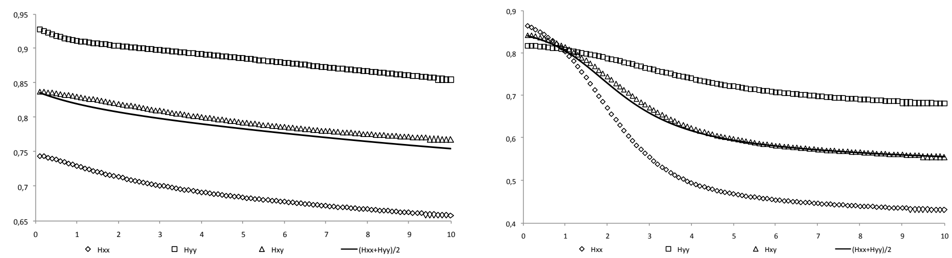

Figure 1: (a) Estimates of $H_{xx}(q)$, $H_{yy}(q)$ and $H_{xy}(q)$ (y-axis) for two ARFIMA processes with $H_{xx} = 0.7$ and $H_{yy} = 0.9$ for different qs (x-axis); (b) Estimates of $H_{xx}(q)$, $H_{yy}(q)$ and $H_{xy}(q)$ (y-axis) for two MBM with $m_0 = 0.2$ and $m_0 = 0.4$, respectively, for different qs (x-axis).

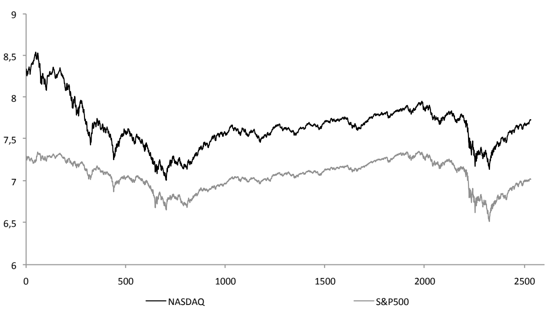

Figure 2: Evolution of logarithmic prices of NASDAQ and S&P500 (y-axis) in time (x-axis) from 1.1.2000 to 31.12.2009.

(a) (b)

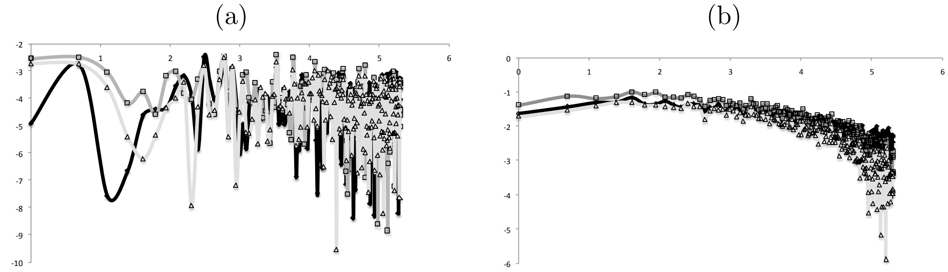

Figure 3: (a) Auto-correlation and cross-correlation function for returns of NASDAQ, S&P500 up to lag 200; (b) Auto-correlation and cross-correlation function for volatility of NASDAQ, S&P500 up to lag 200.

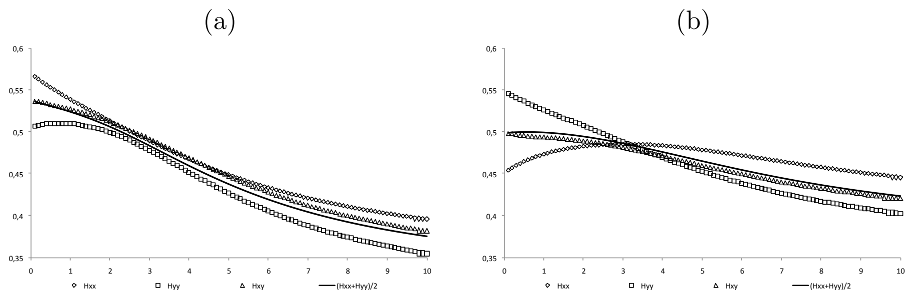

Figure 4: Estimates of $H_{xx}(q)$, $H_{yy}(q)$ and $H_{xy}(q)$ (y-axis) for returns of NASDAQ and S&P500 for $q = 0.1, 0.2, \ldots, 10$ for original (a) and shuffled data (b).

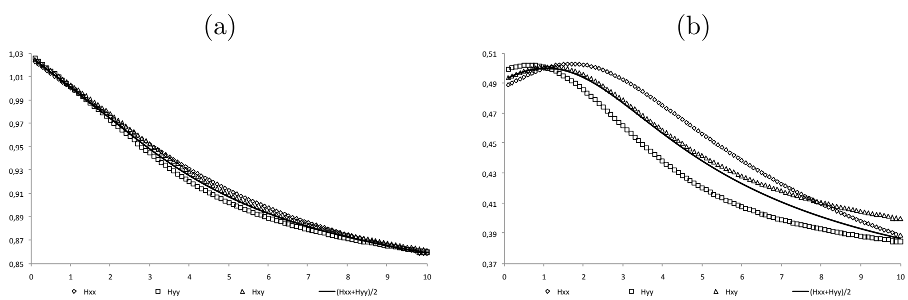

Figure 5: Estimates of $H_{xx}(q)$, $H_{yy}(q)$ and $H_{xy}(q)$ (y-axis) for absolute returns of NASDAQ and S&P500 for $q = 0.1, 0.2, \ldots, 10$ for original data (a) and shuffled data (b).

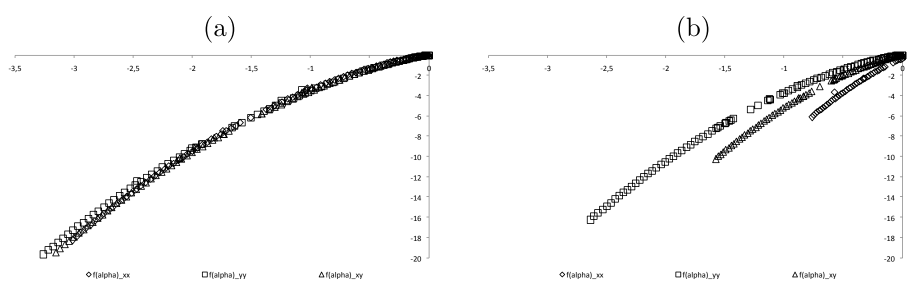

Figure 6: Singularity strengths α (x-axis) and singularity spectra $f(\alpha)$ (y-axis) for returns of NASDAQ, S&P500 and combined for $q = 0.1, 0.2, \ldots, 10$ for original (a) and shuffled data (b).

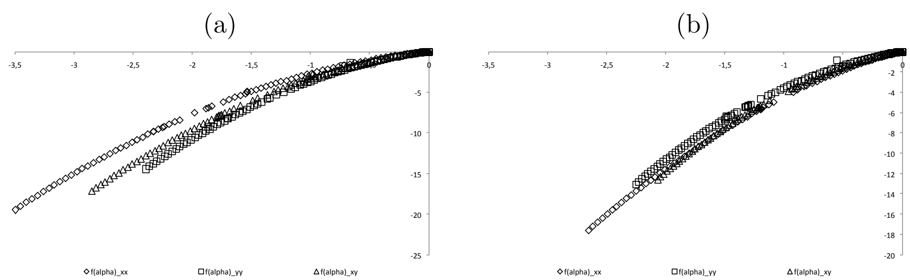

Figure 7: Singularity strengths α (x-axis) and singularity spectra $f(\alpha)$ (y-axis) for absolute returns of NASDAQ, S&P500 and combined for $q = 0.1, 0.2, \ldots, 10$ for original (a) and shuffled (b).